JOURNAL OF APPROXIMATION THEORY 4, 225–229 (1971)

Degeneracy in Mean Rational Approximation

CHARLES B. DUNHAM

Computer Science Department, University of Western Ontario, London 72, Ontario, Canada

Communicated by E. W. Cheney

Received June 10, 1970

In this note we consider the question when can a degenerate rational function be a best mean approximation on an interval $[a, b]$.

Let $R_m^n[a, b]$ be the family of ratios p/q of polynomials p of degree $\leqslant n$ to polynomials q of degree $\leqslant m$ which are positive on $[a, b]$. Let $\int$ denote integration over $[a, b]$ and for $g \in C[a, b]$ define

$$\| g \| = \int | g |.$$

The approximation problem is: Given $f \in C[a, b]$ to find an r^* minimizing $\| f - r \|$ among all elements r of $R_m^n[a, b]$. Any such r^* is called a best (mean, L_1) approximation to f. A proof is given in [1] that every continuous function has a best approximation.

Denote the exact degree of a polynomial p by $\partial(p)$. If a nonzero rational function r is represented as p/q, where p and q have no common factors, then we define the *degeneracy* of r to be $d(r) = \min\{n - \partial(p), m - \partial(q)\}$. The zero rational function has degeneracy m. A rational function with positive degeneracy is said to be *degenerate* (or nonnormal in the terminology of [1, 237]). In the case of Chebyshev approximation, a sufficient condition for f to have r as (unique) best approximation is that $f - r$ alternate $n + m + 1 - d(r)$ times [2, 80]. Thus there exist many functions with degenerate best approximations. On the other hand, Cheney and Goldstein [1, 239] have proved that a degenerate rational function is a best approximation only to itself in the case of mean square approximation. It is, therefore, of interest to see what properties f possesses if it has a degenerate best mean approximation.

If we want to study L_1 approximation with a multiplicative positive weight function w, we can simply define $\int$ to denote integration after multiplication by w; all results will remain true except in the example following (1).

This paper makes heavy use of the techniques of Cheney and Goldstein [1].

PRELIMINARIES

A fundamental role in mean approximation is played by the set

$$Z(r) = \{x : f(x) = r(x), a \leqslant x \leqslant b\}.$$

Let $\sim Z(r)$ stand for $[a, b] \sim Z(r)$.

We will make use of the characterization lemma for linear mean approximation, a proof of which appears in [2, 103].

LEMMA 1. *A necessary and sufficient condition that*

$$\| f - r \| \leqslant \| f - r - \lambda s \| \quad \textit{for all real } \lambda \textit{ is that } \left| \int s \operatorname{sgn}(f - r) \right| \leqslant \int_{Z(r)} | s |.$$

In general, because of the nonconvexity of $R_m{}^n[a, b]$ for $m > 0$, there is no simple test for an element of this space to be a best approximation. In the special case where the approximating function is zero, the above lemma completely answers this question; namely, 0 is a best approximation to f if and only if for all $r \in R_m{}^n[a, b]$,

$$\left| \int_{\sim Z(0)} r \operatorname{sgn}(f) \right| \leqslant \int_{Z(0)} | r |. \tag{1}$$

EXAMPLE. Let $n = 0$, $m = 1$, $a = 0$, $b = 1$. Select f such that $Z(0) \supset [0, 1/4] \cup [3/4, 1]$. We need only consider the case where $r(x) = A/(1 + Bx)$, $A > 0$, $B > -1$. By convexity of r it is easily seen that

$$\int_{1/4}^{3/4} r \leqslant \int_0^{1/4} r + \int_{3/4}^1 r;$$

hence (1) is always satisfied. It follows that 0 is a best approximation to f from $R_1{}^0[0, 1]$ and has degeneracy 1. It is a consequence of Theorem 1 that 0 is a best approximation only to itself if $m \geqslant 2$.

IMPOSSIBILITY OF DEGENERACY TWO

The following lemma is easily proved.

LEMMA 2. *Let g be a positive continuous function on $[a, b]$. If $a < \lambda < \mu < b$, then there exists a quadratic polynomial $s > 0$ such that*

$$\int_\lambda^\mu \frac{g}{s} > \int_W \frac{g}{s}, \qquad W = [a, b] \sim [\lambda, \mu].$$

THEOREM 1. *If the rational function f has degeneracy* 2 *or more, and if r is a best approximation to f, then* $f \equiv r$.

Proof. If $f \not\equiv r$, then $f - r$ is nonzero on an interval $(\lambda, \mu) \subset (a, b)$. Assume that $f - r$ is positive on this interval. Select s as in Lemma 2, with $g = 1/q$. Then

$$r + \frac{\lambda}{qs} = \frac{p}{q} + \frac{\lambda}{qs} = \frac{ps + \lambda}{qs} \in R_m{}^n[a, b].$$

Let $v = 1/(qs)$. Then

$$\left| \int v \cdot \operatorname{sgn}(f - r) \right| \geqslant \int_\lambda^\mu v \cdot \operatorname{sgn}(f - r) - \int_W v \cdot \operatorname{sgn}(f - r),$$

$$\left| \int v \cdot \operatorname{sgn}(f - r) \right| - \int_{Z(r)} | v | \geqslant \int_\lambda^\mu v - \int_W v > 0.$$

It follows by Lemma 1 that r is not a best approximation to f in $\{r + \lambda v\} \subset R_m{}^n[a, b]$, proving the theorem.

DEGENERACY ONE

In the case that r has degeneracy one the theory is less satisfactory. Using arguments similar to those in the proofs of Lemma 2 and Theorem 1, we obtain

LEMMA 3. *Let a positive continuous function g be given, and let* $a < \lambda < b$. *Then there exist first degree polynomials s, t which are positive on* $[a, b]$ *such that*

$$\int_a^\lambda \frac{g}{s} > \int_\lambda^b \frac{g}{s} \qquad \textit{and} \qquad \int_a^\lambda \frac{g}{t} < \int_\lambda^b \frac{g}{t}.$$

THEOREM 2. *Let r be degenerate and let it be a best approximation to f. Then both a and b are limit points of* $Z(r)$.

We now show that if r is a best approximation to f, then $Z(r)$ is of positive measure.

LEMMA 4. *Let* ψ *be a continuous strictly monotonic function on* $[a, b]$. *For every measurable bounded f, the condition*

$$\int f\psi^n = 0, \qquad n = 0, 1, \ldots,$$

implies that $f = 0$ *almost everywhere.*

Proof. The algebra generated by $\{\psi^n : n = 0, 1,...\}$ is dense (with respect to the sup norm) in $C[a, b]$ by Stone's Theorem. The convergence of a sequence with respect to the sup norm on $[a, b]$ implies convergence with respect to the L_1 norm on $[a, b]$; also $C[a, b]$ is dense in $L_1[a, b]$. It follows that the algebra is dense in $L_1[a, b]$. We have, for g in the algebra,

$$\int f^2 = \int fg + \int f(f - g) \leqslant \{\sup |f(x)| : a \leqslant x \leqslant b\} \|f - g\|_1 ;$$

hence $\int f^2 = 0$ and $f = 0$ almost everywhere.

THEOREM 3. *The orthogonal complement of $R_m{}^n[a, b]$, $m > 0$, in the space of bounded measurable functions is the set of functions vanishing almost everywhere.*

Proof. We argue as in the proof of the theorem of Cheney and Goldstein [1, 236–237] with $\psi(x) = x/1$, $p_0 = 1$.

THEOREM 4. *A degenerate element r of $R_m{}^n[a, b]$ cannot be a best approximation to $f \in C[a, b]$ unless $Z(r)$ is a set of positive measure.*

Proof. Suppose $r = p/q$ is degenerate. Let $s(x) = x - \alpha$, $\alpha \notin [a, b]$. For all real λ, $r_\lambda = r + \lambda/qs = (ps + \lambda)/qs \in R_m{}^n[a, b]$. Now suppose r is a best approximation to $f \in C[a, b]$ and $Z(r)$ is a set of measure zero. By Lemma 1,

$$\left| \int \frac{1}{qs} \operatorname{sgn}(f - r) \right| \leqslant \int_{Z(r)} \left| \frac{1}{qs} \right| = 0.$$

This must be true for every $\alpha \notin [a, b]$ and so $\operatorname{sgn}(f - r)/q$ is in the orthogonal complement of $R_1{}^0[a, b]$. Further, $|[\operatorname{sgn}(f - r)]/q| \leqslant 1/\min\{q(x) : a \leqslant x \leqslant b\}$ and $[\operatorname{sgn}(f - r)]/q$ is continuous on $Z(r)$. It follows from Theorem 3 that $[\operatorname{sgn}(f - r)]/q$ vanishes almost everywhere. We have a contradiction and the theorem is proved.

It is a consequence of Theorem 2 or 4 that if r is degenerate and is a best approximation to f, $r - f$ vanishes at an infinite number of points; hence,

COROLLARY. *If r is degenerate, the only analytic function which has r as a best approximation is r itself.*

ACKNOWLEDGMENT

The author would also like to thank the referee for improving Lemma 4 and Theorem 3.

References

1. E. W. Cheney and A. A. Goldstein, Mean Square Approximation by Generalized Rational Functions, *Math. Zeitschr.* **95** (1967), 232–241.
2. J. Rice, "The Approximation of Functions," Vol. 1, Addison-Wesley, Reading, Mass., 1964.

Printed in Belgium